SINGLE-PERIOD STOCHASTIC INVENTORY PROBLEMS WITH QUADRATIC COSTS

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THESIS

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by

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March 1974

Approved for public release; distribution unlimited.

Single-Period Stochastic Inventory

Problems with Quadratic Costs

by

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Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
March 1974

ABSTRACT

Single-period inventory problems such as the newspaper boy problem having quadratic cost functions for either or both shortages and overages are examined to determine the optimal order level under various principles of choice such as minimum expected cost, aspiration level, and minimax regret. Procedures for finding the optimum order levels are developed for both continuous and discrete demand patterns.

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I. INTRODUCTION

In any decision of interest, there are two or more alternative courses of action among which the decision maker must choose. A principle of choice then indicates which alternative is actually to be selected. [2] In inventory problems we are concerned with making optimal decisions with respect to an inventory system and in particular, with making optimal decisions that minimize the total cost of an inventory system. [3]

The purpose of this thesis is to identify and relate certain principles of choice to single-period inventory problems with quadratic cost functions for either or both shortages and overages, in such a manner that the optimal order level could be determined under each principle of choice. Such principles of choice as expectation, aspiration level, and minimax regret will be examined in this paper.

The thesis proceeds by first reviewing the classic single period inventory problem. Procedures for finding the optimal order levels under each of the three principles of choice when the cost functions for both shortages and overages are quadratic are addressed in Chapter III for both the continuous and discrete cases. Chapter IV includes consideration of problems having (1) a linear cost function for overages and a quadratic for

shortages and (2) a constant cost function for shortages and a quadratic for overages. In Chapter V there are conclusions and recommendations for further work. The philosophical implications of inventory theory are beyond the author's intentions and are not considered here.

II. THE CLASSIC SINGLE-PERIOD INVENTORY PROBLEM

This chapter is concerned with single-period inventory problems where the demand for a period is a random variable having either a known or an unknown probability distribution, and costs are linear. This problem has been studied extensively, [1] [2] and forms the basis for the work presented in later chapters.

In an inventory situation in which items are ordered at the beginning of the period, let C_o be the unit cost of shortage (or profit per item as an opportunity cost) and let C_s be the unit cost of surplus at the end of the period. The decision variable is S, the quantity on hand at the beginning of the period. Let demand D be a random variable which denotes the demand during the period, with probability density function f(D) and distribution function F(D). The cost per period is

$$C(S) = \begin{cases} C_s(S-D), & 0 \leq D \leq S, \\ C_o(D-S), & S < D, \end{cases}$$

or alternatively, the profit per period is

$$\pi(S) = \begin{cases} C_oD-C_s(S-D), & 0 \leq D \leq S, \\ C_oS, & S < D. \end{cases}$$

Well known results regarding the optimal order level S_o may be summarized as follows.

A. OPTIMAL SOLUTIONS FOR CASES WHERE F(D) IS ESTIMABLE

To minimize expected cost or maximize expected profit, the optimal order level S_o is chosen so that

$$F(S_o-1) < C_o/(C_s + C_o) < F(S_o) ,$$

if the demand is discrete, or

$$F(S_o) = C_o/(C_s + C_o) ,$$

if the demand is continuous. [1]

To maximize the probability that cost does not exceed the decision maker's aspiration level A, where the demand is continuous, the optimum order level S_o is chosen so that [2]

$$f(S_o - A/C_s) = f(S_o + A/C_o) .$$

B. OPTIMAL SOLUTIONS FOR CASES WHERE D_{max} IS ESTIMABLE BUT F(D) IS NOT

To minimax regret, the optimal order level S_o is chosen such that

$$S_o < C_o(D_{max} + 1)/(C_s + C_o) < S_o + 1 ,$$

if the demand is discrete, or

$$S_o = C_o(D_{max})/(C_s + C_o) ,$$

if the demand is continuous. [2]

The next two chapters will be developed in a similar way as we seek solutions for the quadratic case.

III. SINGLE-PERIOD INVENTORY MODELS HAVING QUADRATIC COST FUNCTIONS

In order to translate a realistic inventory problem into a mathematical problem of minimizing a cost function, both flexible and simple approximations to a wide range of cost relationships are desirable to allow easy mathematical solutions. Consideration of the kinds of costs involved suggests that a U-shaped cost curve is required. [6] For example, the cost of inventory is high when inventory is large, and high also at the other extreme when inventory is so small that there are frequent runouts of inventory. Somewhere between these extremes, the combined costs are at a minimum. With these considerations in view, the cost functions may sometimes be approximated with reasonable accuracy by a positive definite quadratic form. [6]

This chapter is concerned with single-period inventory models having quadratic cost functions for both shortages and overages in connection with both continuous and discrete demand patterns.

A. CONTINUOUS DEMAND PATTERNS

The procedure developed below is a method for finding the optimal order level S_o when the cost functions can be approximated by

$$\text{cost per period, } C(S) = \begin{cases} C_s(S - D)^2 , & 0 \leq D \leq S , \\ C_o(D - S)^2 , & S < D . \end{cases} \qquad (1)$$

1. Minimum Expected Cost Solution

Suppose that the demand for a period is a random variable having a known probability distribution. It is assumed that the demand is continuous. The expected total cost per period of the system is

$$E(C(S)) = \int_{o}^{S} C_s (S-D)^2 \; f(D)dD + \int_{S}^{\infty} C_o (D-S)^2 \; f(D)dD. \qquad (2)$$

To find the optimal order level S_o, the expected cost function is differentiated with respect to S and the results are set equal to zero. This involves differentiation of an integral. It can be shown that if

$$F(t) = \int_{A(t)}^{B(t)} G(t,x),$$

respect to t?

then

$$dF(t)/dt = \int_{A(t)}^{B(t)} \partial G(t,x)dx/\partial t + G[t,B(t)]dB(t)/dt - G[t,A(t)]dA(t)/dt.$$

This is known as Leibniz Rule. Applying this result to equation (2) yields

$$dE(C(S))/dS = 2C_s \int_{o}^{S} (S-D)f(D)dD - 2C_o \int_{S}^{\infty} (D-S)f(D)dD. \qquad (3)$$

Setting the derivative equal to zero and solving leads to

$$(\bar{D} - S_o)/\bar{D}(S_o) - S_o \; F(S_o) = C_o - C_s/C_o, \qquad (4)$$

where $\bar{D}$ = Expection of demand, E(D), and

$$\bar{D}(S_o) = \int_{o}^{S_o} Df(D)dD. \qquad (5)$$

This result (4) agrees with the work of Sirichoke. [5]

To check whether S_o satisfying equation (4) gives a minimum, the second derivative must be examined. From equation (3),

$$d^2E(C(S))/dS^2 = 2C_s\int_0^S f(D)dD + 2C_o\int_S^\infty f(D)dD.$$

Since f(D) is a probability density, the value of the second derivative is always non-negative for non-negative unit costs C_s and C_o. Thus the value of S_o satisfying equation (4) does indeed furnish an optimal solution.

Equation (4) is not tractable in general because of the difficulties imposed by $\bar{D}(S_o)$. For the uniform demand distribution, however, it is possible to obtain S_o and the minimum cost C^* in explicit form. This is illustrated as follows. Let the demand density be

$$f(D) = \begin{cases} 1/D_{max}, & 0 \le D \le D_{max}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\bar{D} = D_{max}/2$ and

$$F(S_o) = \int_0^{S_o} f(D)dD/D_{max} = S_o/D_{max}$$

and

$$\bar{D}(S_o) = \int Df(D)dD/D_{max} = S_o^2/2D_{max}.$$

Substituting these values in equation (4) gives the optimal order quantity

$$S_o = D_{max}/(1 + \sqrt{C_s/C_o}). \qquad (5)$$

With the uniform distribution and S_o, equation (2) gives

$$C^* = (C_s S_o^3 + C_o(D_{max} - S_o)^3)/3D_{max}$$

as the minimum expected cost.

2. <u>Aspiration Level Solutions</u>

It is possibly true that some form of aspiration level principle is the most widely used of all principles in management decision making as alternatives become increasingly expensive to discover. An aspiration level is simply some level of cost which the decision maker desires not to exceed. For the inventory problem we are considering, an aspiration level policy might be expressed as follows. For a given aspiration level, A, select the optimal order level S_o which maximizes the probability that the cost will be equal to or less than A. [2]

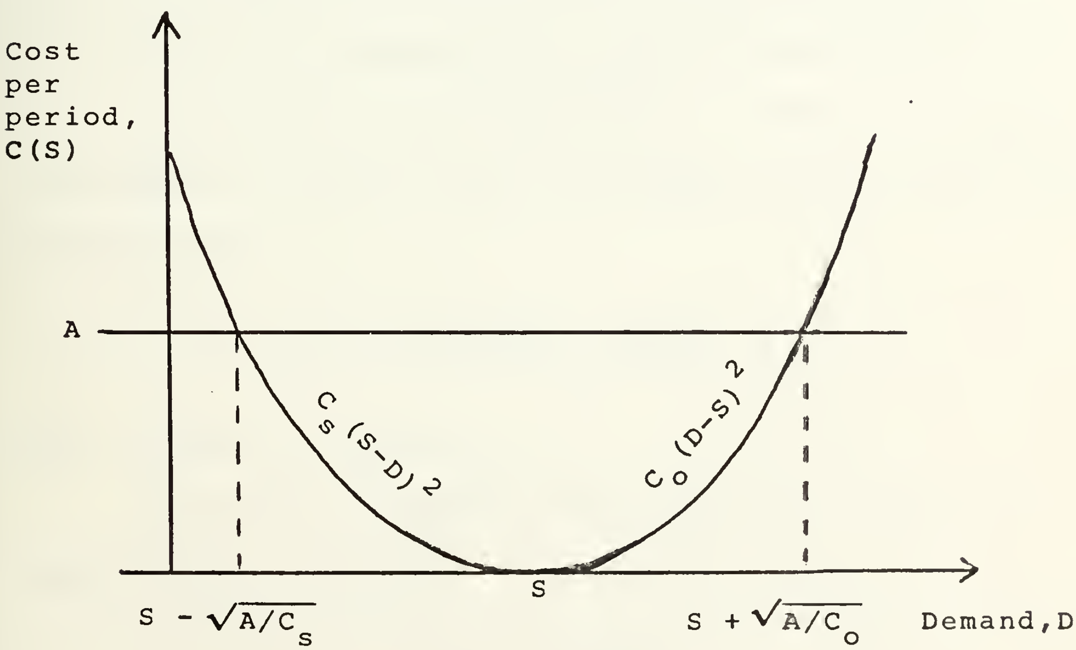

Figure 1. Quadratic Cost Function v.s. Demand, Showing Aspiration Level

From Figure 1, it follows that

$$pr(cost \leq A) = pr(S - \sqrt{A/C_s} \leq D \leq S + \sqrt{A/C_o}) =$$

$$F(S + \sqrt{A/C_o}) - F(S - \sqrt{A/C_s}).$$

It is assumed that the probability density function of the demand is unimodal with respect to the maximum of that function defined on the range of demand. To find the optimal level S_o, pr(cost $\leq$ A) is differentiated with respect to S and the results are set equal to zero:

$$dpr(cost \leq A)/DS = f(S + \sqrt{A/C_o}) - f(S - \sqrt{A/C_s}) = 0 .$$

This leads to

$$f(S_o + \sqrt{A/C_o}) = f(S_o - \sqrt{A/C_s})$$

as our basis for choosing an optimal value of S.

It can be easily shown that if f(D) is unimodal and symmetric about $\bar{D}$ (as, for example, the normal distribution), then

$$S_o = \bar{D} + \frac{1}{2}(\sqrt{A/C_s} - \sqrt{A/C_o}) .$$

This is shown in Figure 2.

If f(D) is unimodal with mode at 0 (as, for example, the exponential distribution), then

$$S_o = \sqrt{A/C_s}$$

as the aspiration level solution. This is shown in Figure 3. We note that in the latter case it is unnecessary to estimate the unit outage cost C_o to achieve an optimal solution.

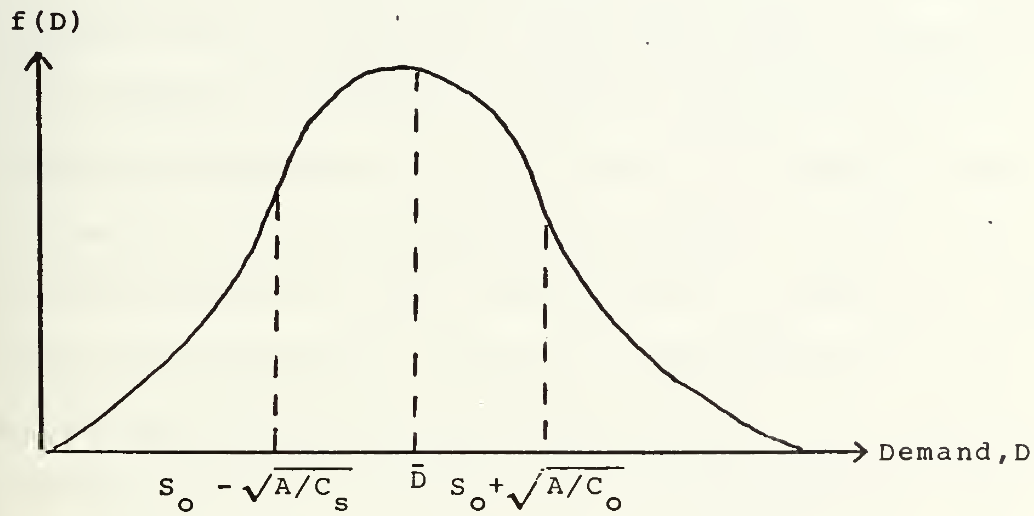

Figure 2. Application of the Aspiration Level Decision Rule to a Unimodal, Symmetric Probability Distribution for Demand

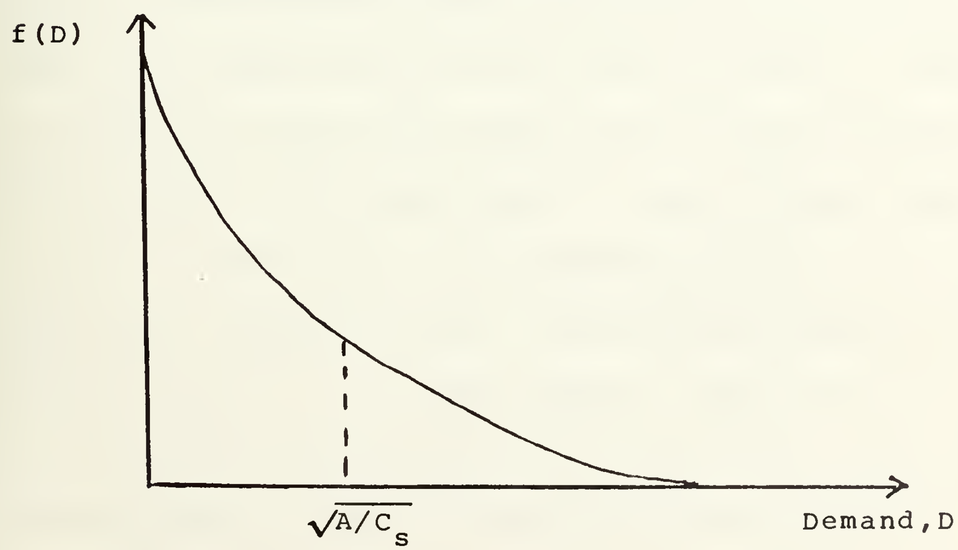

Figure 3. Application of the Aspiration Level Decision Rule to a Unimodel Probability Distribution for Demand Having Mode at 0.

3. Minimax Regret Solutions

A decision for which the analyst elects to consider several possible futures, whose probabilities cannot be estimated is called a decision under uncertainty. [2] When we are unable to estimate f(D), the inventory problem may be treated as a decision under uncertainty. A principle of choice for decisions under uncertainty has been proposed by L. J. Savage, who suggests that a new matrix called a "regret matrix" be computed first. For each possible future combination of demand and order level, the difference should be computed between the actual cost that will occur and the best cost that could be occurred for the future under consideration. This difference is called "regret."[4]

This principle of choice may be appropriate for cases where the maximum demand, D_{max}, is known but f(D) is not. By observing the cost function and bearing in mind that demand takes on any value between 0 and D_{max}, it should be easily seen that the minimum cost will be zero for any given order level S, when demand turns out to be the order level. The maximum surplus cost will be $C_s S^2$, when demand is zero. The maximum shortage cost will be $C_o(D_{max} - S)^2$, when demand turns out to be D_{max}. For any order level S the maximum cost will simply be the maximum of these two quantities and so will the maximum regret. It will be noted that the maximum surplus

cost $C_s S^2$ will increase as order level increases and the maximum shortage cost will decrease as order level increases. The minimax cost will occur when they are equal, thus S_o satisfying the equation

$$C_s S_o^2 = C_o (D_{max} - S_o)^2$$

will be the optimal order level. Solving this for S_o we have

$$S_o = D_{max} / (1 + \sqrt{C_s / C_o})$$

as the minimax regret solution. It should be noted that this result agrees with equation (5) which gives the minimum expected cost solution when demand is uniformly distributed between 0 and D_{max}. Accordingly, for the case where f(D) cannot be estimated, the same result would occur if we had used Laplace's Principle of Insufficient Reason which suggests computing a simple average when probabilities cannot be estimated. [2]

B. DISCRETE DEMAND PATTERNS

Suppose that the demand is a discrete random variable having a known probability distribution. When demand D and order level S are constrained to discrete units 0, u, 2u, ... and p(D) is the probability mass function of demand, then the expected total cost of the system becomes

$$E(C(S)) = C_s \sum_{D=0}^{S} (S-D)^2 p(D) + C_o \sum_{D=S+u}^{\infty} (D-S)^2 p(D). \qquad (6)$$

The necessary conditions for S_o to be the optimal order level are

$$E(C(S_o + u)) - E(C(S_o)) \geq 0 \tag{7}$$

$$E(C(S_o - u)) - E(C(S_o)) \geq 0 . \tag{8}$$

To find the conditions for the system, the differences

$$E(C(S+u)) - E(C(S))$$

and

$$E(C(S-u)) - E(C(S))$$

must be evaluated in general first. Here,

$$E(C(S+u))-E(C(S))=2uC_s \sum_{D=0}^{S} (S-D+ \frac{1}{2}u)p(D)$$

$$-2uC_o \sum_{D=S+u}^{\infty} (D-S- \frac{1}{2}u)p(D)$$

and

$$E(C(S-u))-E(C(S))=-2uC_s \sum_{D=0}^{S} (S-D- \frac{1}{2}u)p(D)$$

$$+2uC_o \sum_{D=S+u}^{\infty} (D-S+ \frac{1}{2}u)p(D) .$$

Applying these results in equations (7) and (8) and solving leads to

$$L(S_o) \leq (C_o - C_s)/C_o \leq U(S_o) , \tag{9}$$

where

$$L(S_o) = (S_o- \frac{1}{2}u -\bar{D})/(F(S_o)(S_o- \frac{1}{2}u)-\bar{D}(S_o)),$$

$$U(S_o) = (S_o+ \frac{1}{2}u -\bar{D})/(F(S_o)(S_o+ \frac{1}{2}u)-\bar{D}(S_o)),$$

and

$$\bar{D}(S_o) = \sum_{D=0}^{S_o} Dp(D) .$$

It can be easily shown that these are also sufficient conditions for S_o to be the optimal order level.

As an example, suppose u = 5, C_s = \$3, C_o = \$30, and the demand for the item for a period is random with probabilities p(0) = 0.05, p(5) = 0.25, p(10) = 0.35, p(15) = 0.30, and p(20)= 0.05. Computation of the values of L(S) and U(S) are summarized in Table I. Since

$$(C_o - C_s)/C_o = 0.9,$$

the optimal order quantity is

$$S_o = 15,$$

because

$$0.857 = L(15) < 0.9 = (C_o - C_s)/C_o < U(15) = 0.98.$$

As a check on this result, E(C(S)) of equation (6) is calculated for all values of S. This is done in the last column of Table I. As expected, the least cost of the system occurs when S = 15.

No rule is given for an aspiration level solution in the discrete demand case because it is straightforward to compute pr (cost $\leq$ A) for each order quantity S, and choose the one which maximizes pr (cost $\leq$ A). To

this, we will consider the previous example. Let A, the aspiration level for cost, be \$1,000. Then the optimal order level S_o is easily found as 15 units from Table II.

TABLE I

Tabulation of L(S) and U(S)

D,S	p(D)	F(S)	D(S)	L(S)	U(S)	E C(S)
0	0.05	0.05	0	102	-62	\$3,862.50
5	0.25	0.30	1.25	15.5	-2.75	1,503.75
10	0.35	0.65	4.75	-22	0.668	408.75
15	0.30	0.95	9.25	0.857	0.98	172.50
20	0.05	1.00	10.25	1.00	1.00	356.25

TABLE II

Computation of pr(cost $\leq$ \$1,000)

S \ D	0	5	10	15	20	pr(cost $\leq$ \$1,000)
p(D)	0.05	0.25	0.35	0.30	0.05	
0	0	750	3000	6750	12000	0.3
5	75	0	750	3000	6750	0.65
10	300	75	0	750	3000	0.95
15	675	300	75	0	750	1.00
20	1200	675	300	75	0	0.95

In a discrete demand pattern, the Savage principle, which suggests computing a regret matrix first, may also be appropriate for the cases where D_{max} is estimable but p(D) is not. For each possible future combination of

demand D and order quantity S, the difference is computed between the actual cost that will occur and the minimum cost that could be occurred. Having completed the regret matrix, the optimal order level is selected which minimizes the maximum regret.

To illustrate this, we will again consider the previous example. A regret matrix is completed first as shown in Table III, and the optimal order level S_o is selected as 15 units, since the regret of that quantity is the minimum.

TABLE III

Regret Matrix

S D	0	5	10	15	20	max regret
0	0	750	3000	6750	12000	12000
5	75	0	750	3000	6750	6750
10	300	75	0	750	3000	3000
15	675	300	75	0	750	750
20	1200	675	300	75	0	1200

So far, we have examined the cases where both shortage and overage cost functions are quadratic. The next chapter will be devoted to examine the cases where cost functions are either a combination of linear and squared terms or of constant and squared terms.

IV. SINGLE PERIOD INVENTORY MODELS HAVING QUADRATIC COST FUNCTION FOR EITHER SHORTAGES OR OVERAGES

Although a mathematical form for describing a realistic inventory problem should be simple and easy to handle, it must be flexible enough to give adequate approximations to a wide variety of situations. With these considerations in view, the cost function may sometimes be approximated by either a combination of linear and squared terms or by a combination of constant and squared terms in controlled and uncontrolled variables. [6] This chapter proceeds by first considering the case where the overage cost function is linear over the region of the number of units overstocked and the shortage cost may be approximated by a quadratic function of the number of units short. The case where the shortage cost is constant and the average cost is quadratic will also be examined.

A. LINEAR COSTS FOR OVERAGES AND QUADRATIC FOR SHORTAGES

The procedure developed below is a method for finding the optimal order level S_o when the cost per period is

$$C(S) = \begin{cases} C_s(S-D), & 0 \leq D \leq S, \\ C_o(D-S)^2, & S < D. \end{cases} \tag{10}$$

1. Minimum Expected Cost Solution

When the demand for a period is a continuous random variable having a known probability distribution,

the expected total cost of the system is

$$E(C(S))=\int_{0}^{S} C_s(S-D)f(D)dD+\int_{S}^{\infty} C_o(D-S)^2 f(D)dD. \qquad (11)$$

Applying Leibniz Rule to equation (11) yields

$$dE(C(S))/dS = C_s F(S) - 2C_o \int_{S}^{\infty} (D-S)f(D)dD. \qquad (12)$$

To find S_o, equation (12) is set equal to zero. This leads to

$$C_s/2C_o = (\bar{D}^c(S_o) - S_o F^c(S_o))/F(S_o) , \qquad (13)$$

where

$$\bar{D}^c(S_o) = \int_{S_o}^{\infty} Df(D)dD$$

and

$$F^c(S_o) = \int_{S_o}^{\infty} f(D)\ dD.$$

To check whether S_o satisfying equation (13) gives a minimum, the second derivative must be examined. From equation (12),

$$d^2E(C(S))/dS^2 = C_s f(S) +2C_o \int_{S}^{\infty} f(D)dD.$$

Since f(D) is a probability density, the value of the second derivative is always positive, thus the order quantity S_o satisfying equation (13) does indeed minimize E(C(S)).

Equation (13) is not tractable in many cases because of the difficulties imposed by $\bar{D}^c(S_o)$. For the uniform

demand distribution it is possible to obtain S_o and the minimum cost C* in explicit form. This is illustrated as follows. Let the demand density be

$$f(D) = \begin{cases} 1/D_{max} , & 0 \leq D \leq D_{max} , \\ 0 & , \text{otherwise.} \end{cases}$$

Then

$$\bar{D}^c(S_o) = \frac{1}{2}D_{max} - (S_o{}^2/2D_{max}),$$

$$F(S_o) = S_o/D_{max} ,$$

and

$$F^c(S_o) = 1 - (S_o/D_{max}).$$

Substituting these values in equation (13) gives the optimal order quantity

$$S_o = D_{max} + (C_s/2C_o) - \sqrt{(C_s D_{max}/C_o) + (C_s/2C_o)^2} \quad . \qquad (14)$$

The minimum cost is

$$C^* = (C_s S_o{}^2/2D_{max}) + (C_o(D_{max} - S_o)((D_{max} - S_o) \quad (D_{max} - S_o)(D_{max} - 4S_o)$$

$$+ 3S_o{}^2))/3D_{max} .$$

When the demand for a period is a discrete random variable having a known probability distribution, the expected total cost of the system is

$$E(C(S)) = C_s \sum_{D=0}^{S_o} (S-D)p(D) + C_o \sum_{D=S+1}^{\infty} (D-S)^2 p(D) \quad . \qquad (15)$$

Using the necessary and sufficient conditions (7) and (8) in Chapter III, we obtain the following results:

$$L(S_o) \leq C_s/2C_o \leq U(S_o) \quad , \tag{16}$$

where

$$L(S_o) = (\bar{D}^c(S_o)+F^c(S_o)(S_o - \tfrac{1}{2}))/F(S_o),$$

$$U(S_o) = (\bar{D}^c(S_o)+F^c(S_o)(S_o + \tfrac{1}{2}))/F(S_o),$$

$$\bar{D}^c(S_o) = \sum_{D=S_o+1}^{\infty} Dp(D),$$

and

$$F^c(S_o) = \sum_{D=S_o+1}^{\infty} p(D).$$

2. <u>Aspiration Level Solutions</u>

Suppose one wishes to keep the cost per period less than or equal to an aspiration level A. One might want to choose order quantity S so that pr(cost $\leq$ A) is maximized.

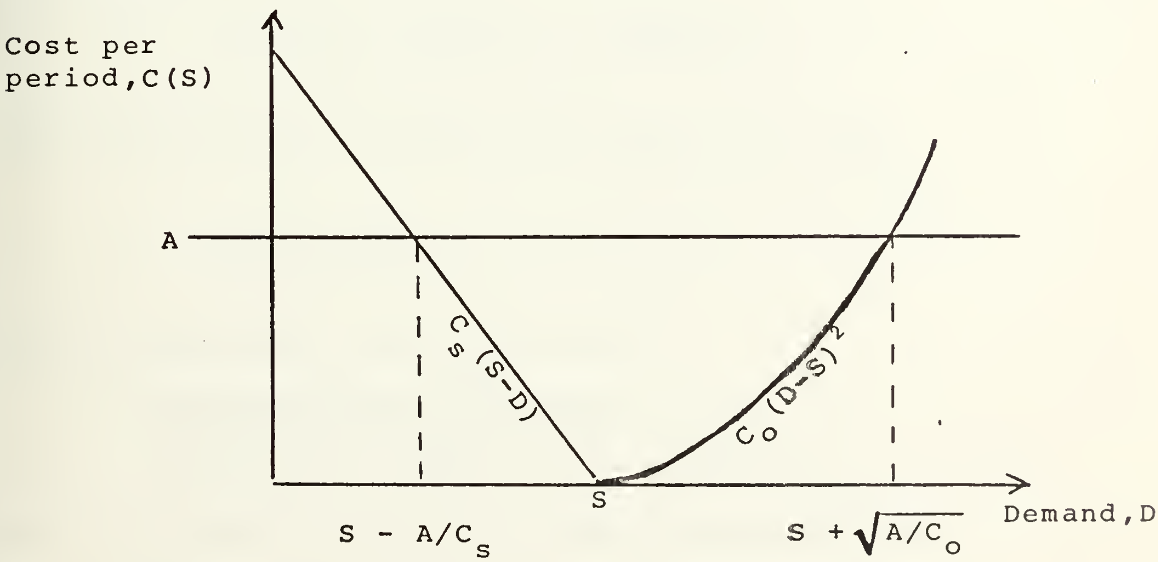

Figure 4. Cost v.s. Demand with Linear Costs for Overages and Quadratic Cost for Shortages, Showing Aspiration Level

From Figure 4, it is seen that pr(cost $\leq$A) is equal to

$$pr((S-(A/C_s)) \leq D \leq (S + \sqrt{A/C_o})),$$

and thus

$$pr(cost \leq A)= F(S+ \sqrt{A/C_o})- F(S-(A/C_s)).$$

It is assumed that the probability density function of the demand is unimodal with respect to the maximum of that function defined on the range of demand. To find the optimal order level S_o, pr(cost $\leq$ A) is differentiated with respect to S. Setting the result equal to zero yields the rule

$$f(S_o+ \sqrt{A/C_o}) = f(S_o - (A/C_s)).$$

From this it can be easily shown that if f(D) is unimodal and symmetric about $\bar{D}$, then

$$S_o = \bar{D} + \frac{1}{2}((A/C_s) -(\sqrt{A/C_o})).$$

Also, if f(D) is unimodal with mode at 0, then

$$S_o = A/C_s ,$$

as the aspiration level solution.*

3. <u>Minimax Regret Solutions</u>

Suppose that the maximum demand, D_{max}, is estimable but f(D) is not and we wish to minimize the

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*This analysis is similar to that of Chapter III.

maximum regret. By the same argument developed in Chapter III, we have

$$\max\{C_s S,\ C_o(D_{max} - S)^2\}$$

as the maximum regret. Since the maximum surplus cost $C_s S$ increases linearly as the order level increases and the maximum shortage cost decreases as the order level increases, S_o satisfying the equation

$$C_s S = C_o(D_{max} - S_o)^2 ,$$

gives the optimal order level. Solving this for S_o we obtain

$$S_o = D_{max} + (C_s/2C_o) - \sqrt{(C_s D_{max}/C_o) + (C_s/2C_o)^2}$$

as the minimax regret solution. This result agrees with equation (14).

If the demand is discrete we obtain the following result:

$$S_o - 1 \leq D_{max} + (C_s/2C_o) - \sqrt{(C_s(D_{max}+1)/C_o) + (C_s/2C_o)^2} \leq S_o$$

as the optimal order level solution.

B. CONSTANT COST FUNCTION FOR SHORTAGES AND QUADRATIC FOR OVERAGES

The procedure developed below is a method for finding the optimal order S_o when the cost per period is

$$C(S) = \begin{cases} C_s(S-D)^2, & 0 \leq D \leq S, \\ K & , S \leq D, \end{cases} \tag{17}$$

where the cost due to understock at the end of the period is a positive constant K, regardless of the number short. Such a function might be appropriate for cases where shortages cause system failure.

1. Minimum Expected Cost Solutions

When demand for a period is a continuous random variable having a known probability distribution which is differentiable everywhere on the domain,

$$E(C(S)) = \int_0^S C_s (S-D)^2 f(D)\,dD + \int_S^\infty K f(D)\,dD. \tag{18}$$

Here,

$$dE(C(S))/dS = 2C_s \int_0^S (S-D) f(D)\,dD - Kf(S). \tag{19}$$

Setting equation (18) equal to zero and solving this leads to

$$K/2C_s = (S_o F(S_o) - \bar{D}(S_o))/f(S_o), \tag{20}$$

where

$$\bar{D}(S_o) = \int_0^{S_o} D f(D)\,dD,$$

and

$$F(S_o) = \int_0^{S_o} f(D)\,dD.$$

Since

$$d^2E(C(S))/dS^2 = 2C_s \int_0^S f(D)\,dD$$

is always positive, S_o satisfying equation (20) does give the optimal solution.

For the uniform demand distribution it is possible to obtain S_o and the minimum cost C^* in explicit form. This is illustrated as below, where the demand density is

$$f(D) = \begin{cases} 1/D_{max}, & 0 \leq D \leq D_{max}, \\ 0 & , \text{otherwise.} \end{cases}$$

Then

$$\bar{D}(S_o) = S_o^2/2D_{max},$$

$$F(S_o) = S_o/2D_{max},$$

and

$$f(S_o) = 1/D_{max}.$$

Substituting these values in equation (20) yields the optimal order quantity

$$S_o = \sqrt{K/C_s}\ .$$

With the uniform distribution and S_o, equation (18) gives

$$C^* = (C_s S_o^3 + 3K(D_{max} - S_o))/3D_{max}\ .$$

When the demand for a period is a discrete random variable having a known probability distribution p(D), then

$$E(C(S)) = \sum_{D=0}^{S} C_s (S-D)^2 p(D) + \sum_{D=S+1}^{\infty} Kp(D).$$

Using the necessary and sufficient conditions (7) and (8) in Chapter III, we get

$$L(S_o) \leq K/2C_s \leq U(S_o)\ ,$$

where

$$L(S_o) = (F(S_o)(S_o - \frac{1}{2}) - \bar{D}(S_o))/p(S_o),$$

and

$$U(S_o) = (F(S_o)(S_o + \frac{1}{2}) - \bar{D}(S_o))/p(S_o).$$

2. Aspiration Level Solutions

Our cost function is shown in Figure 5. For an aspiration level solution we will be concerned with the relative magnitudes of constant shortage cost K and aspiration level A.

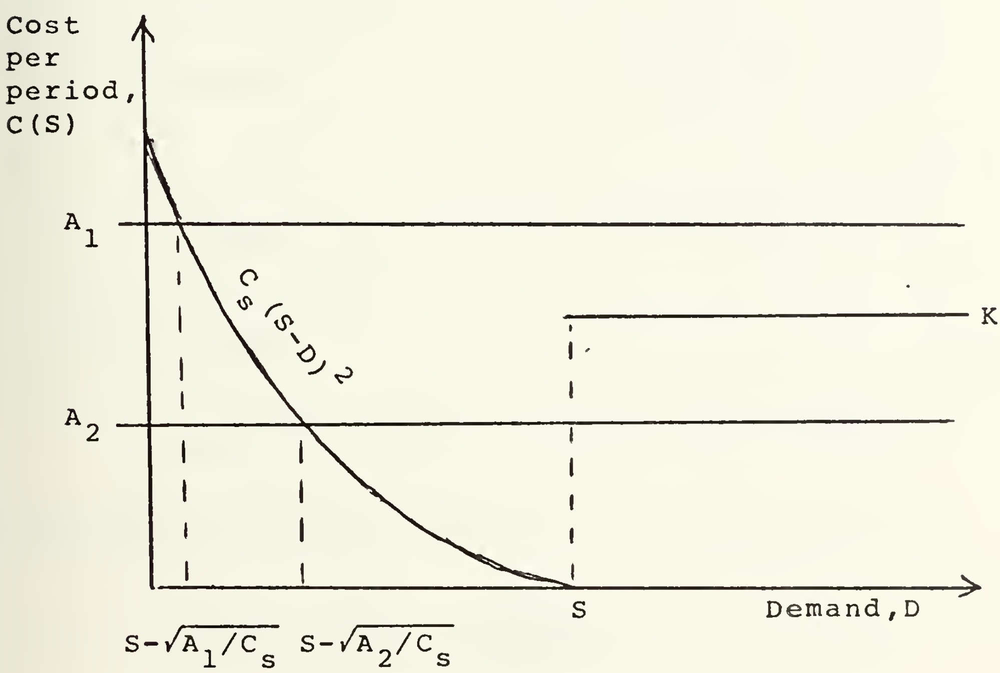

Figure 5. Cost v.s. Demand with Constant Costs for Shortages and Quadratic for Surplus, Showing Aspiration Level

When $A \geq K$, it is easily seen from Figure 5 that

$$pr(\text{cost} \leq A_1) = pr(S - \sqrt{A_1/C_s} \leq D < \infty)$$

and thus

$$pr(\text{cost} \leq A_1) = 1 - F(S - \sqrt{A_1/C_s}),$$

and the probability $pr(\text{cost} \leq A_1)$ is maximized when $F(S - \sqrt{A_1/C_s})$ is at the minimum, which will occur when S takes any value such that

$$0 \leq S_o = \sqrt{A_1/C_s} \ .$$

When $A < K$, we see from Figure 5 that

$$pr(\text{cost} \leq A_2) = pr(S - \sqrt{A_2/C_s} \leq D \leq S),$$

or

$$pr(\text{cost} \leq A_2) = F(S) - F(S - \sqrt{A_2/C_s}) \ .$$

It follows that

$$dpr(\text{cost} \leq A_2)/dS = f(S) - f(S - \sqrt{A_2/C_s}).$$

Setting the result equal to zero yields the rule

$$f(S_o) = f(S_o - \sqrt{A_2/C_s}).$$

Thus it can be shown that if f(D) is unimodal and symmetric about $\bar{D}$ (e.g., normal), then

$$S_o = \bar{D} + \tfrac{1}{2}\sqrt{A_2/C_s} \ .$$

Also, if f(D) is unimodal with mode at 0, then

$$S_o = \sqrt{A_2/C_s} ,$$

as the aspiration level solution. This result agrees with the aspiration level solution to the quadratic surplus cost function.

3. Minimax Regret Solutions

Suppose that D_{max} is estimable but f(D) is not and we wish to minimize the maximum regret. By inspection of the cost function (17) and assuming a continuous demand, it is easy to compute the maximum costs for a given S as shown in Table IV, and it will be noted that this is equivalent to the regret matrix.

TABLE IV

A Computation of Costs for Various Demands

Demand, D	Surplus Cost	Shortage Cost	Max Cost
0	$C_s S^2$	0	$C_s S^2$
S	0	0	0
D>S	0	K	K

From Table IV, it is clear that the maximum regret is simply

$$\max\{C_s S^2, K\} .$$

Since the maximum surplus cost $C_s S^2$ increases with S while the maximum shortage cost is unchanged, we want S_o such that

$$C_s S_o^2 \leq K ,$$

and solving for S_o we have

$$0 \leq S_o \leq \sqrt{K/C_s}$$

as the optimal order level solution.

If the demand is discrete we will minimax regret for any S such that $C_s S^2 \leq K$, thus S_o is any quantity satisfying the inequality

$$0 \leq S_o \leq \sqrt{K/C_s} .$$

We have seen so far how the principles of choice may be applied to single-period inventory problems for decision making under risk and under uncertainty. Along these lines some suggestions for additional work are made in the next chapter.

V. CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER WORK

We have reviewed the classic single-period inventory problem in the second chapter and in a similar way, procedures for finding the optimal order level under various principles of choice have been developed when the cost functions are quadratic for both shortages and surplus in Chapter III. In Chapter IV, other interesting single-period inventory problems have been examined.

When demand is a continuous random variable, Leibniz Rule becomes a very useful tool in differentiating of an integral of which the integrand is quadratic and it is noted that non-negative demand distribution combined with non-negative shortage and surplus costs assures the second derivative of the expected cost function always positive and thus the minimum expected cost is easily found by simply setting the first derivative equal to zero and solving for the optimal order level S_o. When demand is a discrete random variable, the finite difference inequations become very useful tools to find the minimum expected cost solution. In a discrete demand case, no rule is given for an aspiration level solution. because it is straightforward to compute the probability that cost is less than or equal to a given aspiration level A, for each order quantity S, and choose the one

which maximizes pr (cost $\leq$ A). In a continuous demand case, the assumption of the demand density being a uni-modal is a crucial thing to achieve an aspiration level solution. Also, we have noted that the general rules for the minimum expected cost solution are not tractable in many cases because of the difficulties imposed by the integrals $\bar{D}(S_o)$ or $\bar{D}^c(S_o)$. We have seen, however, that for the uniform demand distribution, it is possible to find the optimal solution, which agrees with that of the minimax regret, when the maximum demand is estimable. The use of more than one principle of choice has resulted in several possible optimal solutions.

In a variety of actual contexts the cost function might be better approximated by a piece-wise linear or exponential function. It must also be pointed out that, in decisions under risk, not only the expectation, but perhaps also the variance and other parameters of the distribution should be taken into account. For example, if two different order levels have the same expected cost, then the one with the smaller variance of the expected cost will be chosen. It is recommended that further work along these lines be conducted.

This thesis has examined a number of interesting and possibly realistic extensions of the classic single-period inventory model, both in terms of variations in the cost function and in the applications of various principles of choice. It is hoped that this work will be useful to those interested in this kind of inventory problem.

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Single-Period Stochastic Inventory Problems with Quadratic Costs		5. TYPE OF REPORT & PERIOD COVERED Master's Thesis March 1974
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Moon Ho Song		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		12. REPORT DATE March 1974
		13. NUMBER OF PAGES 36
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) Naval Postgraduate School Monterey, California 93940		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Single-Period Inventory
Quadratic Costs
Principles of Choice

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Single-period inventory problems such as the newspaper boy problem having quadratic cost functions for either or both shortages and overages are examined to determine the optimal order level under various principles of choice such as minimum expected cost, aspiration level, and minimax regret. Procedures for finding the optimum order levels are developed for both continuous and discrete demand patterns.

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